

# ASYMPTOTICS OF WILLMORE MINIMIZERS WITH PRESCRIBED SMALL ISOPERIMETRIC RATIO

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ABSTRACT. We consider surfaces in  $\mathbb{R}^3$  of type  $\mathbb{S}^2$  which minimize the Willmore functional with prescribed isoperimetric ratio. In [21] Schygulla proved the existence of smooth minimizers. In the singular limit when the isoperimetric ratio converges to zero, he showed convergence to a double round sphere in the sense of varifolds. Here we give a full blowup analysis of this limit, showing that the two spheres are connected by a catenoidal neck. Besides its geometric interest, the problem was studied as a simplified model in the theory of cell membranes, see e.g. [2].

## 1. INTRODUCTION

The isoperimetric ratio of a smooth embedding  $f : \mathbb{S}^2 \rightarrow \mathbb{R}^3$  is defined as

$$(1.1) \quad \sigma(f) = 6\sqrt{\pi} \frac{\mathcal{V}(f)}{\mu(f)^{3/2}} \in (0, 1].$$

The area  $\mu(f)$  and enclosed volume  $\mathcal{V}(f)$  are given by

$$(1.2) \quad \mu(f) = \int_{\mathbb{S}^2} Jf \, d\mu_{\mathbb{S}^2} \quad \text{and} \quad \mathcal{V}(f) = \frac{1}{3} \int_{\mathbb{S}^2} \langle f, \vec{n} \rangle \, d\mu_f,$$

where  $\vec{n} : \mathbb{S}^2 \rightarrow \mathbb{S}^2$  is the exterior unit normal. The normalization is chosen such that  $\sigma(\mathbb{S}^2) = 1$ . It is an interesting geometric problem to find natural representatives among surfaces with prescribed isoperimetric ratio  $\sigma \in (0, 1)$ . In [21] Schygulla introduced a variational approach by minimizing the Willmore energy

$$\mathcal{W}(f) = \frac{1}{4} \int_{\mathbb{S}^2} |\vec{H}|^2 \, d\mu_f.$$

He proved that the infimum among topological spheres is attained by a smooth minimizer, for any prescribed isoperimetric ratio  $\sigma \in (0, 1]$ . An existence result for higher genus surfaces was proved more recently by Keller, Mondino and Rivière, assuming that the infimum of the energy satisfies certain inequalities [10]. In any case minimizers solve the Euler Lagrange equation

$$(1.3) \quad \frac{1}{2} (\Delta_g H + |A^\circ|^2 H) = \Lambda \sigma(f) \left( \frac{1}{\mathcal{V}(f)} + \frac{3}{2\mu(f)} H \right).$$

Here the left hand side is the Euler Lagrange operator of the Willmore energy, and the right hand side is the Euler Lagrange operator of the isoperimetric ratio. By Alexandrov's theorem the isoperimetric constraint is nondegenerate and hence the Lagrange multiplier  $\Lambda \in \mathbb{R}$  is well-defined, with the only exception of round spheres.

In his paper [21] Schygulla also studies sequences of minimizers with isoperimetric ratio converging to zero. He shows that up to translations and dilations, the sequence converges in the varifold sense to a round sphere of multiplicity two [21, Thm. 2]. In the present paper we study this singular limit more precisely, and obtain the following asymptotic results. Here we denote by  $N, S$  the north and south pole of  $\mathbb{S}^2$  and by  $\pi_N, \pi_S$  the stereographic projections from the poles.

**Theorem 1.1.** *Let  $f_k : \mathbb{S}^2 \rightarrow \mathbb{R}^3$  be conformally parametrized  $\mathcal{W}$ -minimizers for prescribed isoperimetric ratio  $\sigma(f_k) = \sigma_k \rightarrow 0$ . After conformal reparametrization, scaling and translating, and passing to a subsequence, the following hold:*

- (1)  $\mu(f_k) = 1$ ,
- (2)  $f_k$  converges locally smoothly on  $\mathbb{S}^2 \setminus \{N\}$  to a conformal immersion  $f^0 : \mathbb{S}^2 \rightarrow \mathbb{R}^3$  with  $f^0(N) = 0$ .
- (3) There exist  $r_k \rightarrow 0$  such that the sequence  $f_k^1(z) = f_k \circ \pi_S^{-1}(r_k z)$  converges locally smoothly on  $\mathbb{R}^2$  to a conformal immersion  $f^1 : \mathbb{R}^2 \rightarrow \mathbb{R}^3$  with  $f^1(\infty) = 0$ .
- (4) Both  $f^0, f^1$  are conformal equivalences to the same round sphere  $\mathbb{S}$  of area  $1/2$ .
- (5) There exist  $t_k, \lambda_k \rightarrow 0$ , such that the sequence  $f_k^2(z) = \lambda_k^{-1} f_k \circ \pi_S^{-1}(t_k z)$  converges locally smoothly on  $\mathbb{R}^2 \setminus \{0\}$  to a conformally parametrized catenoid  $f^2 : \mathbb{R}^2 \setminus \{0\} \rightarrow \mathbb{R}^3$ , with the origin and the center of  $\mathbb{S}$  on the symmetry axis.
- (6) We have the energy identity  $\lim_{k \rightarrow +\infty} \int_{\mathbb{S}^2} |A_{f_k}|^2 d\mu_{f_k} = \sum_{i=0,1,2} \int |A_{f^i}|^2 d\mu_{f^i}$ .
- (7) We have  $\lim_{k \rightarrow \infty} (\log t_k : \log \lambda_k : \log r_k) = (1 : 1 : 2)$ , and  $\lim_{k \rightarrow \infty} \Lambda_k / \lambda_k$  exists.

The geometric problem we address here is partially motivated by a model for cell membranes due to Helfrich [7]. In the case of topological spheres the Helfrich energy becomes

$$F_{SC} = \frac{\varkappa}{2} \int_{\mathbb{S}^2} (H - C_0)^2 d\mu_g + 4\pi \varkappa_G,$$

where  $\varkappa, \varkappa_G$  and  $C_0$  are constants. The Helfrich energy essentially reduces to the Willmore energy when the spontaneous curvature  $C_0$  is zero. Although this simplified case may not correspond to real membranes, it was studied by several authors. In the axially symmetric case the minimizers are described by ordinary differential equations which were studied by numerical approximations, see e.g. [3, 17, 2]. In particular Berndt, Lipowsky and Seifert [2] distinguished three kinds of shapes depending on the isoperimetric ratio  $\sigma$ , which they call the reduced volume: the prolate-dumbbell, the oblate-discocyte and the stomatocyte type. The stomatocyte parameter range is indicated as  $0 < \sigma \leq 0.591$ , and it is noted that a neck develops as  $\sigma \rightarrow 0$ . Here we provide a rigorous analysis for this neck formation, without assuming axial symmetry.

## 2. BUBBLE LIMITS OF MINIMIZERS $f_k$ FOR ISOPERIMETRIC RATIO $\sigma_k \rightarrow 0$

Let  $\Sigma$  be a Riemann surface. We denote by  $W_{conf}^{2,2}(\Sigma, \mathbb{R}^3)$  the set of  $f \in W_{loc}^{2,2}(\Sigma, \mathbb{R}^3)$ , such that in any conformal chart the metric has the form  $g_{ij} = e^{2u} \delta_{ij}$  where  $u \in L_{loc}^\infty$  [11]. More generally, a branched conformal immersion on  $\Sigma$  is a map  $f \in W_{conf}^{2,2}(\Sigma \setminus \mathcal{S}, \mathbb{R}^3)$  where  $\mathcal{S}$

is finite and

$$\mu_g(\Omega \setminus \mathcal{S}) + \int_{\Omega \setminus \mathcal{S}} |A|^2 d\mu_g < \infty \quad \text{for any } \Omega \subset \Sigma.$$

Any branched conformal immersion is in  $W_{loc}^{2,2}(\Sigma, \mathbb{R}^3)$ . At each  $p \in \Sigma$  the map has multiplicity  $m_p \in \mathbb{N}_0$  and behaves like  $z^{m_p}$  in a weak sense.

In this section we summarize results from [4] about the convergence of a sequence  $f_k : \mathbb{S}^2 \rightarrow \mathbb{R}^3$  of (branched) conformal immersions with bounded Willmore energy and area. We first recall the notion of a bubble: a sequence  $z_k \in \mathbb{C}$ ,  $r_k > 0$ , with  $z_k \rightarrow 0$  and  $r_k \rightarrow 0$  is called a bubble of  $f_k$  at  $p \in \mathbb{S}^2$ , if the following holds: in some local conformal coordinates with  $p$  corresponding to  $z = 0$  the sequence  $\hat{f}_k(z) = f_k(z_k + r_k z)$  converges locally  $W^{2,2}$ -weakly on  $\mathbb{C}$ , away from finitely many points, to a nonconstant map  $f : \mathbb{C} \rightarrow \mathbb{R}^3$ . It follows that when composed with stereographic projection, the blowup limit  $f$  is a branched conformal immersion from  $\mathbb{S}^2$  to  $\mathbb{R}^3$  with finite area and finite Willmore energy. This notion of a bubble is clearly independent of the coordinates. Two bubbles  $(z_k^i, r_k^i)$ ,  $i = 1, 2$ , are different if they concentrate at different points  $p_i \in \mathbb{S}^2$ , or if their scales are different in the sense that

$$\frac{r_k^1}{r_k^2} + \frac{r_k^2}{r_k^1} + \frac{|z_k^1 - z_k^2|}{r_k^1 + r_k^2} \rightarrow \infty.$$

To construct a limit of the sequence  $f_k$  we first observe the diameter of  $f_k(\mathbb{S}^2)$  is bounded. In the case when  $f_k$  is a smooth immersion, the diameter estimate was proved by Simon [20] using a monotonicity formula. For a  $W^{2,2}$ -conformal immersion one can approximate by smooth immersions to get the result. If the  $f_k$  are merely branched conformal immersions one considers the image 2-varifold as in [14]. Now after translating we can assume

$$f_k(\mathbb{S}^2) \subset \overline{B_R(0)} \quad \text{for all } k.$$

Together with the Willmore energy bound, this shows that  $f_k$  belongs to the class  $\mathcal{F}^2(\mathbb{S}^2, g_{\mathbb{S}^2}, R)$  defined in [4, Sec.2]. Passing to a subsequence, we can also arrange that

$$\|A_{f_k}\|_{g_k}^2 d\mu_{g_k} \rightarrow \alpha \quad \text{in } C^0(\mathbb{S}^2)'.$$

Depending on a constant  $\varepsilon_0 > 0$ , which will be chosen appropriately, the finite concentration set is defined by

$$\mathcal{S} = \{p \in \mathbb{S}^2 : \alpha(\{p\}) \geq \varepsilon_0\}.$$

Proposition 2.1 in [4] says that there is a subsequence such that  $f_k \rightarrow f^0$  weakly in  $W^{2,p}$  for any  $p < 2$  and strongly in  $W^{1,q}$  for any  $q < \infty$ , away from  $\mathcal{S}$ . In particular we can assume  $Df_k \rightarrow Df^0$  pointwise almost everywhere. By Hélein's convergence result, see e.g. [11, Thm. 5.1], we get for a subsequence the following alternative:

- (a) either  $u_k$  is locally bounded in  $\mathbb{S}^2 \setminus \mathcal{S}$ , and  $f_k$  converges  $W^{2,2}$ -weakly away from  $\mathcal{S}$  to a branched conformal immersion  $f : \mathbb{S}^2 \rightarrow \mathbb{R}^3$ .
- (a) or  $u_k \rightarrow -\infty$  and  $f_k \rightarrow \text{const.}$  locally uniformly on  $\mathbb{S}^2 \setminus \mathcal{S}$ .

To rule out the case that the limit  $f^0$  is constant, we will apply a general bubbling result from [4, Thm. 2.8]. In the case of surfaces of type  $\mathbb{S}^2$  it says the following.

**Lemma 2.1.** *Let  $f_k : \mathbb{S}^2 \rightarrow \mathbb{R}^3$  be a sequence of (branched) conformal immersions with bounded Willmore energy and area. Then there exists a branched conformal immersion  $f^0 : \mathbb{S}^2 \rightarrow \mathbb{R}^3$  (possibly constant) and a set of different bubbles  $(z_k^i, r_k^i)$ ,  $1 \leq i \leq N$  (possibly empty), such that the following holds for a subsequence:*

$$(2.1) \quad \lim_{k \rightarrow \infty} \int_{\mathbb{S}^2} d\mu_{f_k} = \sum_{i=0}^N \int_{\mathbb{S}^2} d\mu_{f^i},$$

$$(2.2) \quad \limsup_{k \rightarrow \infty} \mathcal{W}(f_k) \leq \sum_{i=0}^N \mathcal{W}(f^i).$$

**Lemma 2.2.** *Let  $f_k : \mathbb{S}^2 \rightarrow \mathbb{R}^3$  be minimizers for the Willmore energy with prescribed isoperimetric ratio  $\sigma_k = \sigma(f_k) \rightarrow 0$ . After composing with suitable Möbius transformations, the following holds:*

- (a)  $f_k$  converges to a conformal equivalence  $f^0 : \mathbb{S}^2 \rightarrow \mathbb{S}$ , where  $\mathbb{S}$  is a round sphere of area  $1/2$  passing through the origin.
- (b)  $f_k$  has a bubble  $f^1 : \mathbb{C} \rightarrow \mathbb{S}$  which extends to a conformal equivalence  $f^1 : \hat{\mathbb{C}} \rightarrow \mathbb{S}$  with orientation opposite to  $f^0$ .

Moreover,  $f_k$  has no further bubbles.

*Proof.* In [21] Schygulla proved that up to translations  $f_k(\mathbb{S}^2) \rightarrow 2\mathbb{S}$  as varifolds, where  $\mathbb{S}$  is as stated. The monotonicity formula in [20] implies that the convergence is also in Hausdorff distance. We claim that when applying Lemma 2.1, the map  $f^0$  can be assumed nonconstant. Otherwise by (2.2) there is at least one bubble  $f^1 : \mathbb{S}^2 \rightarrow \mathbb{R}^3$ , having area  $\mu(f^1) > 0$ . Up to rotation, the bubble concentrates at the north pole, which means

$$f_k(\pi_S^{-1}(z_k + r_k z)) \rightarrow f^1(z) \quad W^{2,2}\text{-weakly locally in } \mathbb{R}^2, \text{ away from a finite set.}$$

The maps  $\pi_S^{-1} \circ A_k \circ \pi_S$ ,  $A_k(z) = z_k + r_k z$ , extend to Möbius transformations of  $\mathbb{S}^2$ , and

$$f_k(\pi_S^{-1} \circ A_k \circ \pi_S) \rightarrow f^1 \circ \pi_S \neq \text{const.}$$

Thus we obtained a nonconstant limit  $f^0 := f^1 \circ \pi_S : \mathbb{S}^2 \rightarrow \mathbb{S}$ , proving the claim. Now  $f^0$  is a branched conformal immersion, in particular  $f^0 \in W^{2,2} \cap W^{1,\infty}(\mathbb{S}^2, \mathbb{R}^3)$ . We compute in local complex coordinates on  $\mathbb{S}^2$ , writing  $\langle \partial_i f^0, \partial_j f^0 \rangle = e^{2u_0} \delta_{ij}$ ,

$$\langle \Delta f^0, \partial_j f^0 \rangle = \partial_i \langle \partial_i f, \partial_j f \rangle - \frac{1}{2} \partial_j |\partial_i f|^2 = \partial_i (e^{2u_0} \delta_{ij}) - \frac{1}{2} \partial_j (2e^{2u_0}) = 0.$$

So  $f^0 : \mathbb{S}^2 \rightarrow \mathbb{S}$  is weakly harmonic, and hence smooth by standard regularity theory [9, Thm. 2.5.1]. Orienting  $\mathbb{S}^2$  and  $\mathbb{S}$  by their exterior normals, we conclude that  $f^0 : \mathbb{S}^2 \rightarrow \mathbb{S}$  is either holomorphic or anti-holomorphic. This implies  $\mu(f^0) = d|\mathbb{S}|$  for  $d \in \{1, 2\}$ .

Assume by contradiction  $d = 2$ . Then  $\mu(f^0) = 1 = \mu(f_k)$ , which implies  $f_k \rightarrow f^0$  in  $W^{1,2}(\mathbb{S}^2, \mathbb{R}^3)$  by conformality. But the volume functional is continuous under  $W^{1,2}$ -convergence of uniformly bounded maps, and so  $\mathcal{V}(f^0) = \lim_{k \rightarrow \infty} \mathcal{V}(f_k) = 0$ , a contradiction. Thus we have  $d = 1$  and  $f^0 : \mathbb{S}^2 \rightarrow \mathbb{S}$  is a conformal automorphism. Now by (2.1) there exists a bubble  $f^1 : \mathbb{S}^2 \rightarrow \mathbb{S}$ . The argument above shows that this is again

a conformal equivalence. Now for any collection of different bubbles we have the lower semicontinuity

$$\mu(f^0) + \sum_{i=1}^N \mu(f^i) \leq \liminf_{k \rightarrow \infty} \mu(f_k) = 1.$$

Since  $\mu(f^0) + \mu(f^1) = 1$ , a further bubble would have zero area and again be conformal, hence constant. This is ruled out by the definition of bubble.

To show that  $f^0$  and  $f^1$  are oppositely oriented, we compute with  $\omega = \frac{1}{3}x \lrcorner dx^1 \wedge dx^2 \wedge dx^3$

$$\int_{\mathbb{S}^2 \setminus B_\varrho(N)} f_k^* \omega + \int_{D_{\frac{1}{\varrho}}(0)} \hat{f}_k^* \omega = \int_{\mathbb{S}^2 \setminus A_{k,\varrho}} f_k^* \omega = \mathcal{V}(f_k) - \int_{A_{k,\varrho}} f_k^* \omega,$$

where  $A(k, \varrho) = \mathbb{S}^2 \setminus (B_\varrho(N) \cup \pi_S^{-1}(D_{\frac{r_k}{\varrho}}(z_k)))$ . The error term is estimated by

$$\left| \int_{A_{k,\varrho}} f_k^* \omega \right| \leq C \int_{A_{k,\varrho}} d\mu_{f_k} = C \left( 1 - \int_{\mathbb{S}^2 \setminus B_\varrho(N)} d\mu_{f_k} - \int_{D_{\frac{1}{\varrho}}(0)} d\mu_{\hat{f}_k} \right).$$

Letting  $k \rightarrow \infty$  we obtain using  $\lim_{k \rightarrow \infty} \mathcal{V}(f_k) = 0$

$$\begin{aligned} \left| \int_{\mathbb{S}^2 \setminus B_\varrho(N)} (f^0)^* \omega + \int_{D_{\frac{1}{\varrho}}(0)} (f^1)^* \omega \right| &\leq \limsup_{k \rightarrow \infty} \left| \int_{A_{k,\varrho}} f_k^* \omega \right| \\ &\leq C \left( 1 - \int_{\mathbb{S}^2 \setminus B_\varrho(N)} d\mu_{f^0} - \int_{D_{\frac{1}{\varrho}}(0)} d\mu_{f^1} \right). \end{aligned}$$

Letting  $\varrho \rightarrow 0$  proves our claim.  $\square$

### 3. ESTIMATES FOR CRITICAL POINTS WITH PRESCRIBED ISOPERIMETRIC RATIO

The first variation of the Willmore energy at  $f : \Sigma \rightarrow \mathbb{R}^3$  in direction  $\phi$  is given by

$$\begin{aligned} \delta \mathcal{W}(f) \phi &= \frac{1}{2} \int_{\Sigma} \langle \vec{H}, \Delta_g \phi \rangle d\mu_g \\ &\quad - \int_{\Sigma} g^{\alpha\beta} g^{\lambda\mu} \langle \vec{H}, A_{\alpha\lambda} \rangle \langle \partial_\beta f, \partial_\mu \phi \rangle d\mu_g \\ &\quad + \frac{1}{4} \int_{\Sigma} |\vec{H}|^2 g^{\alpha\beta} \langle \partial_\alpha f, \partial_\beta \phi \rangle d\mu_g. \end{aligned} \tag{3.1}$$

Writing  $\vec{H} = H\vec{n}$  and  $\phi = \varphi\vec{n}$ , we get by partial integration for  $\phi$  having compact support

$$\delta \mathcal{W}(f) \phi = \frac{1}{2} \int_{\Sigma} (\Delta_g H + |A^\circ|^2 H) \varphi d\mu_g. \tag{3.2}$$

The first variation of the isoperimetric ratio is

$$\begin{aligned} \delta \sigma(f) \phi &= \sigma(f) \left( \frac{1}{\mathcal{V}(f)} \int_{\Sigma} \langle \phi, \vec{n} \rangle d\mu_g - \frac{3}{2\mu(f)} \int_{\Sigma} \langle df, d\phi \rangle_g d\mu_g \right) \\ &= \sigma(f) \left( \frac{1}{\mathcal{V}(f)} \int_{\Sigma} \varphi d\mu_g + \frac{3}{2\mu(f)} \int_{\Sigma} H \varphi d\mu_g \right). \end{aligned} \tag{3.3}$$

The Euler Lagrange operators, i.e. the  $L^2$  gradients, of the functionals are thus

$$(3.4) \quad \nabla \mathcal{W}(f) = \frac{1}{2}(\Delta_g \vec{H} + |A^\circ|^2 \vec{H}),$$

$$(3.5) \quad \nabla \sigma(f) = \sigma(f) \left( \frac{1}{\mathcal{V}(f)} \vec{n} + \frac{3}{2\mu(f)} \vec{H} \right).$$

For  $\lambda > 0$  we have the scaling  $\nabla \mathcal{W}(\lambda f) = \lambda^{-3} \nabla \mathcal{W}(f)$  and also  $\nabla \sigma(\lambda f) = \lambda^{-3} \nabla \sigma(f)$ .

Note that  $\vec{H} = H \vec{n}$  with  $\vec{n}$  the exterior normal implies  $H = -2$  for  $\mathbb{S}^2$ .

**Lemma 3.1.** *Let  $f : \mathbb{S}^2 \rightarrow \mathbb{R}^3$  be a smoothly embedded Willmore minimizer with prescribed isoperimetric ratio  $\sigma(f) \in (0, 1)$ . Then*

$$(3.6) \quad \delta \mathcal{W}(f) = \Lambda \delta \sigma(f) \quad \text{for some } \Lambda \in \mathbb{R}.$$

*Proof.* Assume that  $\delta \sigma(f)$  is identically zero. Then

$$(3.7) \quad H \equiv -\frac{2\mu(f)}{3\mathcal{V}(f)}.$$

By Alexandrov's or by Hopf's theorem,  $f$  parametrizes a round sphere which contradicts the assumption  $\sigma(f) < 1$ . Thus  $\delta \sigma(f) \phi \neq 0$  for some  $\phi$ , and the claim follows by the Lagrange multiplier rule.  $\square$

Scaling to  $\mu(f) = 1$  yields  $\sigma = 6\sqrt{\pi} \mathcal{V}(f)$ , and equation (3.6) becomes

$$(3.8) \quad \frac{1}{2}(\Delta_g H + |A^\circ|^2 H) = \Lambda \sigma(f) \left( \frac{6\sqrt{\pi}}{\sigma(f)} + \frac{3}{2} H \right) = \frac{3\Lambda}{2} (4\sqrt{\pi} + \sigma H).$$

For  $f : D \rightarrow \mathbb{R}^3$  conformal the first variation formula as given in [19] is

$$(3.9) \quad \delta \mathcal{W}(f) \phi = \int_D \langle Q[f], d\phi \rangle \quad \text{for} \quad Q[f] = \frac{1}{2}(\nabla \vec{H} - \frac{3}{2} H \nabla \vec{n} + \frac{1}{2} \vec{H} \times \nabla^\perp \vec{n}),$$

$$(3.10) \quad \delta \sigma(f) \phi = \int_D \langle R[f], \phi \rangle \quad \text{for} \quad R[f] = \frac{\sigma(f)}{\mathcal{V}(f)} \partial_1 f \times \partial_2 f - \frac{3\sigma(f)}{2\mu(f)} \Delta f.$$

The scaling is  $Q[\lambda f] = \lambda^{-1} Q[f]$  and also  $R[\lambda f] = \lambda^{-1} R[f]$ . The Euler Lagrange equation (3.6), scaled to  $\mu(f) = 1$ , has the form

$$(3.11) \quad \operatorname{div} Q[f] = \frac{3\Lambda}{2} (4\sqrt{\pi} \partial_1 f \times \partial_2 f - \sigma \Delta f).$$

By the results in Appendix 1 estimates for  $|\nabla^m f|$  follow once the boundedness of the Lagrange multiplier  $\Lambda$  is established.

**Lemma 3.2.** *Let  $f_k : D \rightarrow \mathbb{R}^3$  be smooth conformal immersions, satisfying for  $\sigma_k \rightarrow 0$*

$$\delta \mathcal{W}(f_k) \phi = \frac{3\Lambda_k}{2} \int_D \langle 4\sqrt{\pi} \vec{n} + \sigma_k \vec{H}_{f_k}, \phi \rangle d\mu_{f_k} \quad \text{for all } \phi \in C_c^\infty(D, \mathbb{R}^3).$$

*Let  $g_k = e^{2u_k} \delta$  be the induced metrics, and assume*

$$(3.12) \quad \|A_{f_k}\|_{L^2(D)} + \|u_k\|_{W^{1,2} \cap L^\infty(D)} \leq C < \infty.$$

*Then the sequence  $\Lambda_k$  is bounded.*

*Proof.* The first variation of the Willmore energy is estimated using (3.1) and (3.12) by

$$|\delta\mathcal{W}(f_k, \phi)| \leq C \left( \|\Delta\phi\|_{L^2(D)} + \|D\phi\|_{L^\infty(D)} \right).$$

On the right hand side of the equation, we first note using (3.12)

$$\int_D |H_{f_k}| d\mu_{f_k} \leq \|H_{f_k}\|_{L^2(g_k)} \mu(f_k)^{1/2} \leq C.$$

By (3.12)  $f_k$  is bounded in  $W^{2,2}(D, \mathbb{R}^n)$ . After passing to a subsequence, we have  $f_k \rightarrow f$  strongly in  $W^{1,2}(D, \mathbb{R}^n)$  and  $\mu(f) = \lim_{k \rightarrow \infty} \mu(f_k) \in (0, \infty)$ . Now

$$\pm \int_D \langle \vec{n}_k, \phi \rangle d\mu_{f_k} = \int_D \langle \partial_1 f_k \times \partial_2 f_k, \phi \rangle dx \rightarrow \int_D \langle \partial_1 f \times \partial_2 f, \phi \rangle dx.$$

For suitable  $\phi$  the right hand side is nonzero. The bound for  $\Lambda_k$  follows.  $\square$

**Remark.** In the case  $\sigma_k \rightarrow \sigma > 0$  the argument is modified as follows. We have

$$\int_D \langle \vec{H}_{f_k}, \phi \rangle d\mu_{f_k} = \int_D \langle \Delta f_k, \phi \rangle dx \rightarrow \int_D \langle \Delta f, \phi \rangle dx.$$

Thus one gets a bound for  $\Lambda_k$  unless  $f$  satisfies the equation

$$-\Delta f = \frac{4\sqrt{\pi}}{\sigma} \partial_1 f \times \partial_2 f,$$

i.e.  $f$  is a conformally parametrized with constant mean curvature  $H_f = \pm 4\sqrt{\pi}/\sigma$ .

#### 4. CONSTRUCTION OF THE CATENOID NECK

Let  $f_k : \mathbb{S}^2 \rightarrow \mathbb{R}^3$  be the sequence of minimizers for  $\sigma(f_k) = \sigma_k \rightarrow 0$  from Lemma 2.2. Thus  $f_k \rightarrow f^0$  weakly in  $W^{2,2}$  away from the north pole, and  $f_k$  has a bubble  $f^1$  concentrating at the north pole. The maps  $f^0, f^1 : \mathbb{S}^2 \rightarrow \mathbb{S}$  are conformal diffeomorphisms, with opposite orientations. In particular

$$\mathcal{W}(f^0) = \mathcal{W}(f^1) = 4\pi \quad \text{and} \quad \int_{\mathbb{S}^2} |A_{f^0}|^2 d\mu_{f^0} = \int_{\mathbb{S}^2} |A_{f^1}|^2 d\mu_{f^1} = 8\pi.$$

On the other hand we know that

$$\mathcal{W}(f_k) \rightarrow 8\pi \quad \text{and} \quad \int_{\mathbb{S}^2} |A_{f_k}|^2 d\mu_{f_k} = 4\mathcal{W}(f_k) - 8\pi \rightarrow 24\pi.$$

In the following we study  $f_k$  near the north pole using the conformal coordinates given by the projection  $\pi_S$ . For convenience we denote  $f_k \circ \pi_S^{-1}$  again by  $f_k$ . Thus putting  $f_k^1(z) = f_k(z_k + r_k z)$ , we have  $f_k^1 \rightarrow f^1(z)$  locally smoothly on  $\mathbb{C}$  up to a finite set, according to Lemma 2.2. By rotating we can arrange that  $z_k = 0$ , which means

$$f_k^1(z) = f_k(r_k z) \rightarrow f^1(z) \quad \text{for all } z \in \mathbb{C}.$$

Now by lower semicontinuity of the Willmore energy, we have

$$\begin{aligned} 4\pi &= \lim_{r \searrow 0} \mathcal{W}(f^0, \mathbb{S}^2 \setminus D_r) \leq \lim_{r \searrow 0} \liminf_{k \rightarrow \infty} \mathcal{W}(f_k, \mathbb{S}^2 \setminus D_r), \\ 4\pi &= \lim_{r \searrow 0} \mathcal{W}(f^1, D_{\frac{1}{r}}) \leq \lim_{r \searrow 0} \lim_{k \rightarrow \infty} \mathcal{W}(f_k^1, D_{\frac{1}{r}}) = \lim_{r \searrow 0} \lim_{k \rightarrow \infty} \mathcal{W}(f_k, D_{\frac{r_k}{r}}). \end{aligned}$$

We conclude that

$$(4.1) \quad \lim_{r \rightarrow 0} \lim_{k \rightarrow \infty} \mathcal{W}(f_k, D_r \setminus D_{\frac{r_k}{r}}) = 0, \quad \text{and similarly}$$

$$(4.2) \quad \lim_{r \rightarrow 0} \lim_{k \rightarrow \infty} \int_{D_r \setminus D_{\frac{r_k}{r}}} |A_{f_k}|^2 d\mu_{f_k} \leq 8\pi.$$

To find the neck we fix some number  $\delta > 0$  and chose  $t_k \in [\frac{r_k}{\delta}, \delta]$  such that

$$\text{diam } f_k(\partial D_{t_k}) = \min_{t \in [\frac{r_k}{\delta}, \delta]} \text{diam } f_k(\partial D_t) =: \lambda_k > 0.$$

Then we have  $\lambda_k \rightarrow 0$ , in fact for any  $r \in (0, \delta]$  we have

$$\limsup_{k \rightarrow \infty} \lambda_k \leq \limsup_{k \rightarrow \infty} \text{diam } f_k(\partial D_r) = \text{diam } f^0(\partial D_r) \rightarrow 0 \text{ for } r \rightarrow 0.$$

This implies that also  $t_k \rightarrow 0$ : if we had  $t_k \rightarrow r > 0$  for a subsequence, then we would get

$$\lambda_k = \text{diam } f_k(\partial D_{t_k}) \rightarrow \text{diam } f^0(\partial D_r) > 0.$$

Similarly  $t_k/r_k \rightarrow \infty$ , because if  $t_k/r_k \rightarrow R < \infty$  for a subsequence then

$$\lambda_k = \text{diam } f_k(\partial D_{t_k}) = \text{diam } f_k(\partial D_{r_k \cdot \frac{t_k}{r_k}}) \rightarrow \text{diam } f^1(\partial D_R) > 0.$$

Now we introduce the rescalings

$$f_k^2 : \mathbb{C} \rightarrow \mathbb{R}^3, \quad f_k^2(z) = \frac{f_k(t_k z) - f_k(t_k)}{\lambda_k}.$$

After passing to a subsequence, we have convergence of the measures

$$\alpha_k^2 = \mu_{f_k^2} |A_{f_k^2}|^2 \rightarrow \alpha \quad \text{in } C_c^0(\mathbb{R}^2)'.$$

We show that  $\alpha_k^2$  do not concentrate away from the origin. Assume by contradiction that  $\varepsilon_1 = \alpha(\{p\}) > 0$  for some  $p \in \mathbb{C} \setminus \{0\}$ . We chose  $R > 0$  such that  $\alpha(\overline{D_R(p)} \setminus \{p\}) < \varepsilon_1/2$ , and define

$$r_k^2 = \inf\{r > 0 : \alpha_k^2(D_r(z)) \geq \frac{\varepsilon_1}{2} \text{ for some } z \in \overline{D_R(p)}\}.$$

As  $\alpha(\{p\}) = \varepsilon_1$  we have  $r_k^2 \rightarrow 0$  as  $k \rightarrow \infty$ . Let  $z_k \in \overline{D_R(p)}$  be a point where the infimum is attained. Then  $z_k \rightarrow p$ , and we have

$$\frac{\varepsilon_1}{2} = \alpha_k^2(D_{r_k^2}(z_k)) \geq \alpha_k^2(D_{r_k^2}(z)) \quad \text{for all } z \in \overline{D_R(p)}.$$

We rescale the sequence again, by putting

$$f_k^3(z) = \frac{f_k^2(z_k + r_k^2 z) - f_k^2(z_k)}{\lambda_k^2} \quad \text{where } \lambda_k^2 = \text{diam } f_k^2(z_k + [0, r_k^2]).$$

We compute

$$\begin{aligned} f_k^3(z) &= \frac{1}{\lambda_k^2} \left( \frac{f_k(t_k(z_k + r_k^2 z)) - f_k(t_k)}{\lambda_k} - \frac{f_k(t_k z_k) - f_k(t_k)}{\lambda_k} \right) \\ &= \frac{f_k(t_k z_k + t_k r_k^2 z) - f_k(t_k z_k)}{\lambda_k \lambda_k^2}. \end{aligned}$$



Furthermore, using that  $f_k^2(z_k + \varrho) = \frac{1}{\lambda_k}(f_k(t_k(z_k + \varrho)) - f_k(t_k))$ , we see that

$$\lambda_k \lambda_k^2 = \text{diam } f_k(t_k z_k + [0, t_k r_k^2]) \leq C.$$

The last step used the diameter estimate in [20], given the Willmore energy bound and the area bound for  $f_k$ . Now by local curvature estimates, see Lemma 6.1, and the convergence as in Theorem 7.1, we obtain

$$f_k^3 \rightarrow f^3 \quad \text{locally smoothly in } \mathbb{C}.$$

We conclude that  $f^3 : \mathbb{C} \rightarrow \mathbb{R}^3$  is a complete minimal embedding with total Gauß curvature at least  $-4\pi$ . By the known classification of minimal immersions  $f^3$  is either an Enneper surface or a plane. But the Enneper is not embedded, and the plane is also ruled out because

$$\frac{\varepsilon_1}{2} = \lim_{k \rightarrow \infty} \alpha_k^2(D_{r_k^2}(z_k)) = \lim_{k \rightarrow \infty} \int_{D_1(0)} |A_{f_k^3}|^2 d\mu_{f_k^3} = \int_{D_1(0)} |A_{f^3}|^2 d\mu_{f^3}.$$

This contradiction shows that the sequence  $f_k^2 : \mathbb{C} \rightarrow \mathbb{R}^3$  has no curvature concentrations away from the origin. Now by the small energy estimates, see Lemma 6.1 or [19], we conclude that

$$f_k^2 \rightarrow f^2 \quad \text{locally smoothly on } \mathbb{C}.$$

By the normalization  $\text{diam } f_k^2(\partial D) = 1$  we have that  $f^2$  is nonconstant, hence it is a complete minimal embedding. Moreover we have again

$$\int_{\mathbb{C}} K_{f^2} d\mu_{f^2} \geq -4\pi.$$

We also know that  $\text{diam } f^2(\partial D_t) \geq 1$  for all  $t > 0$ , and hence  $f^2$  must have ends at zero and infinity. Altogether this implies that  $f^2$  parametrizes a catenoid, and we have the energy identity

$$(4.3) \quad 24\pi = \lim_{k \rightarrow \infty} \int_{\mathbb{S}^2} |A_{f_k}|^2 d\mu_{f_k} = \int_{\mathbb{S}^2} |A_{f^0}|^2 d\mu_{f^0} + \int_{\mathbb{C}} |A_{f^1}|^2 d\mu_{f^1} + \int_{\mathbb{C} \setminus \{0\}} |A_{f^2}|^2 d\mu_{f^2}.$$

The convergence of the sequence  $f_k$  is subsumed as follows:

$$\begin{aligned} f_k(z) &\text{ converges to } f^0 : \mathbb{S}^2 \xrightarrow{\sim} \mathbb{S} \text{ locally smoothly on } \mathbb{S}^2 \setminus \{N\}, \\ f_k^1(z) = f_k(r_k z) &\text{ converges to } f^1 : \hat{\mathbb{C}} \xrightarrow{\sim} \mathbb{S} \text{ locally smoothly on } \mathbb{C}, \\ f_k^2(z) = \frac{1}{\lambda_k}(f_k(t_k z) - f_k(t_k)) &\text{ converges to a catenoid } f^2 \text{ smoothly on } \mathbb{C} \setminus \{0\}. \end{aligned}$$

Moreover  $f_k(t_k) \rightarrow f^0(0) = f^1(\infty)$ , and  $f^0(0) = 0$  is arranged by initial translation.

## 5. ASYMPTOTICS OF THE LIMIT

We finally analyse the asymptotics of the sequence  $f_k$ . Our first goal is as follows.

**Lemma 5.1.** *The parameters  $t_k$ ,  $\lambda_k$  and  $r_k$  satisfy*

$$\lim_{k \rightarrow \infty} (\log t_k : \log \lambda_k : \log r_k) = (1 : 1 : 2).$$

*Proof.* We again consider  $f_k : \mathbb{C} \rightarrow \mathbb{R}^3$  using the chart  $\pi_S^{-1} : \mathbb{C} \rightarrow \mathbb{S}^2 \setminus \{S\}$ . The induced metric  $(g_k)_{ij} = e^{2u_k} \delta_{ij}$  satisfies the Liouville equation

$$-\Delta u_k = K_{f_k} e^{2u_k} \quad \text{on } \mathbb{C}.$$

Let  $u_k^*(r) = \int_0^{2\pi} u_k(r, \theta) d\theta$ . It follows that

$$-(r(u_k^*)')' = - \int_0^{2\pi} r \left( \partial_r^2 u_k + \frac{\partial_r u_k}{r} \right) d\theta = - \int_0^{2\pi} r \Delta u_k(r, \theta) d\theta = \int_0^{2\pi} r K_{f_k} e^{2u_k} d\theta.$$

Thus for any  $\varrho \in [\frac{t_k}{\delta}, \delta]$ , we can estimate recalling our description of convergence

$$\begin{aligned} \sup_{\varrho \in [\frac{t_k}{\delta}, \delta]} |\delta(u_k^*)'(\delta) - \varrho(u_k^*)'(\varrho)| &\leq \int_{\frac{t_k}{\delta}}^{\delta} \int_0^{2\pi} |K_{f_k}| e^{2u_k} r d\theta dr \\ &\leq \frac{1}{4\pi} \int_{D_\delta \setminus D_{\frac{t_k}{\delta}}} |A_{f_k}|^2 d\mu_{f_k} \\ &< \varepsilon \quad \text{for all } k \geq k(\varepsilon, \delta). \end{aligned}$$

This implies for  $k \geq k(\varepsilon, \delta)$

$$\sup_{\varrho \in [\frac{t_k}{\delta}, \delta]} \varrho |(u_k^*)'(\varrho)| \leq \delta |(u_k^*)'(\delta)| + \varepsilon.$$

Now we have

$$(u_k^*)'(r) = \int_0^{2\pi} \partial_r u_k(r, \theta) d\theta \rightarrow \int_0^{2\pi} \partial_r u(r, \theta) d\theta = (u^*)'(r),$$

where  $u(r, \theta)$  is the conformal factor of the smooth equivalence  $f^0 : \mathbb{S}^2 \rightarrow \mathbb{S}$ . We compute using the divergence theorem

$$(u^*)'(\delta) = \int_0^{2\pi} \partial_r u(\delta, \theta) d\theta = \int_{\partial D_\delta} \langle \nabla u, \partial_r \rangle ds = \frac{1}{2\pi\delta} \int_{D_\delta} \Delta u dx dy \rightarrow 0 \quad \text{as } \delta \rightarrow 0.$$

Thus  $\delta |(u^*)'(\delta)| < \varepsilon$  for  $\delta > 0$  sufficiently small, and we obtain

$$\sup_{\varrho \in [\frac{t_k}{\delta}, \delta]} \varrho |(u_k^*)'(\varrho)| < 2\varepsilon \quad \text{for } k \geq k(\varepsilon, \delta).$$

Integration yields

$$\sup_{r \in [\frac{t_k}{\delta}, \delta]} |u_k^*(r) - u_k^*(\delta)| \leq 2\varepsilon \log \frac{\delta}{r} \quad \text{for } k \geq k(\varepsilon, \delta).$$

Using again  $u_k^*(\delta) \rightarrow u^*(\delta)$  we finally get for all  $r \in [\frac{t_k}{\delta}, \delta]$

$$|u_k^*(r) - u^*(\delta)| \leq 2\varepsilon \left( 1 + \log \frac{\delta}{r} \right) \quad \text{if } \delta < \delta(\varepsilon), k \geq k(\delta, \varepsilon).$$

Next, we prove that

$$(5.1) \quad \text{osc}_{D_{2r} \setminus D_r} u_k \leq C \quad \text{for any } r \in [\frac{t_k}{\delta}, \delta].$$

Consider  $f_{k,r}(z) = f_k(rz)$  where  $r \in [\frac{t_k}{\delta}, \delta]$ . From Corollary 2.4 in [11], for any  $p \in \partial D_{\frac{3}{2}}$  there exists a solution to the equation

$$-\Delta \omega_{k,r} = K_{f_{k,r}} e^{2u_{k,r}} \quad \text{in } D_1(p),$$

satisfying the estimates

$$\|\omega_{k,r}\|_{L^\infty(D_1(p))} + \|\nabla \omega_{k,r}\|_{L^2(D_1(p))} \leq C.$$

Since  $u_{k,r} - \omega_{k,r}$  is harmonic, we have the estimate

$$\text{osc}_{D_{\frac{1}{2}}(p)}(u_{k,r} - \omega_{k,r}) \leq C \|\nabla(u_{k,r} - \omega_{k,r})\|_{L^q(D_1(p))} \quad \text{for any } q \in (0, 2).$$

Now let  $f : \mathbb{S}^2 \rightarrow \mathbb{R}^3$  be a conformal immersion with induced metric  $g = e^{2u} g_{\mathbb{S}^2}$ , where  $g_{\mathbb{S}^2}$  is the round metric normalized to  $\mu_{g_{\mathbb{S}^2}}(\mathbb{S}^2) = 1$ . By Liouville we have

$$-\Delta_g u = K_f e^{2u} - 4\pi.$$

As  $\|K_f e^{2u} - 4\pi\|_{L^1} \leq 16\pi$ , we have the estimate

$$(5.2) \quad r^{q-2} \|\nabla_{g_{\mathbb{S}^2}} u\|_{L^q(B_r(p))} \leq C(q) \quad \text{for any } q \in (0, 2).$$

Applying (5.2) we see that

$$\|\nabla u_{k,r}\|_{L^q(D_1(p))} = r^{q-2} \|\nabla u_k\|_{L^q(D_r(p))} \leq C(q).$$

By a covering argument, we get  $\text{osc}_{D_2 \setminus D_1} u_{k,r} \leq C$ . But  $\text{osc}_{D_2 \setminus D_1} u_{k,r} = \text{osc}_{D_{2r} \setminus D_r} u_k$ , thus the oscillation bound (5.1) is proved.

We now come to comparing  $t_k$  with  $\lambda_k$ . We know that  $\frac{t_k}{\lambda_k} \nabla f_k(t_k z)$  converges to  $\nabla f^2$  smoothly, hence we have for  $k$  sufficiently large

$$2 \sup_{\partial D_{\frac{1}{\delta}}} |\nabla f^2| \geq \frac{t_k}{\lambda_k} e^{u_k(t_k z)} \geq \frac{1}{2} \inf_{\partial D_{\frac{1}{\delta}}} |\nabla f^2| > 0.$$

By (5.1) we have  $|u_k(t_k z) - u_k^*(z)| \leq C$  for  $|z| = \frac{1}{\delta}$ . Hence, we get

$$-C \leq \log \frac{t_k}{\lambda_k} + u_k^*\left(\frac{t_k}{\delta}\right) \leq C.$$

Then

$$-C + (1 + \epsilon) \log t_k \leq \log \lambda_k \leq C + (1 - \epsilon) \log t_k.$$

Letting first  $k \rightarrow \infty$  and then  $\epsilon \rightarrow 0$ , we get

$$\lim_{k \rightarrow \infty} \frac{\log t_k}{\log \lambda_k} = 1.$$

To estimate  $r_k$  we consider  $\tilde{f}_k(z) = f_k(\frac{r_k}{z})$ . We compute

$$\tilde{f}_k(r_k z) = f_k\left(\frac{1}{z}\right) \rightarrow f_0\left(\frac{1}{z}\right), \quad \tilde{f}_k(z) \rightarrow f^1\left(\frac{1}{z}\right).$$

Putting  $\tilde{t}_k = r_k/t_k$  we further obtain

$$\frac{\tilde{f}_k(\tilde{t}_k z)}{\lambda_k} = \frac{f_k(t_k z)}{\lambda_k} \rightarrow f^2\left(\frac{1}{z}\right).$$

Using the arguments from above, we get

$$\lim_{k \rightarrow \infty} \frac{\log \tilde{t}_k}{\log \lambda_k} = 1, \quad \text{hence} \quad \lim_{k \rightarrow \infty} \frac{\log r_k}{\log \lambda_k} = 2.$$

□

Now we address the asymptotics of the multipliers  $\Lambda_k$ . For a conformal immersion we have in local coordinates the first variation formulae, see (3.9),

$$\begin{aligned} \delta \mathcal{W}(f)\phi &= \int_D \langle Q[f], d\phi \rangle \quad \text{where } Q[f] = \frac{1}{2}(\nabla \vec{H} - \frac{3}{2}H \nabla \vec{n} + \frac{1}{2}\vec{H} \times \nabla^\perp \vec{n}), \\ \delta \sigma(f)\phi &= \int_D \langle R[f], \phi \rangle \quad \text{where } R[f] = \frac{\sigma(f)}{\mathcal{V}(f)} \partial_1 f \times \partial_2 f - \frac{3\sigma(f)}{2\mu(f)} \Delta f. \end{aligned}$$

The scaling is  $Q[\lambda f] = \lambda^{-1}Q[f]$  and also  $R[\lambda f] = \lambda^{-1}R[f]$ . The Euler Lagrange equation (3.6), scaled to  $\mu(f) = 1$ , has the form

$$(5.3) \quad \operatorname{div} Q[f] = \frac{3\Lambda}{2} (4\sqrt{\pi} \partial_1 f \times \partial_2 f - \sigma \Delta f) := \frac{3\Lambda}{2} S[f].$$

For simplicity we assume  $0 \notin f^2(\mathbb{C} \setminus \{0\})$ . Put  $I(y) = \frac{y}{|y|^2}$  and  $I_k(y) = I(\frac{y}{\lambda_k}) = \lambda_k I(y)$ . Note that  $I_k \circ I_k = \operatorname{id}$ . From the previous section, we compute using the assumption  $f_k(t_k) = 0$

$$I_k(f_k(t_k z)) = I\left(\frac{f_k(t_k z)}{\lambda_k}\right) \rightarrow I(f^2(z)) \quad \text{for all } z \in \mathbb{R}^2.$$

We need to compute the equation satisfied by the maps  $F_k = I_k \circ f_k$ . We compute

$$\begin{aligned} \delta \mathcal{W}(F_k, \phi) &= \frac{d}{dt} \mathcal{W}(F_k + t\phi)|_{t=0} \\ &= \frac{d}{dt} \mathcal{W}(I_k \circ (F_k + t\phi))|_{t=0} \\ &= \delta \mathcal{W}(I_k(F_k), D_{F_k} I_k(\phi)) \\ &= \delta \mathcal{W}(f_k, D_{F_k} I_k(\phi)) \\ &= \Lambda_k \delta \sigma(f_k, D_{F_k} I_k(\phi)) \end{aligned}$$

Then we have

$$\begin{aligned} \int_D \langle Q[F_k], d\phi \rangle dx &= \Lambda_k \sigma(f_k) \int_D \langle R[f_k], D_{F_k} I_k(\phi) \rangle dx \\ &= \frac{3}{2} \Lambda_k \int_D \langle S[f_k], D_{F_k} I_k(\phi) \rangle dx. \end{aligned}$$

Since  $I_k(F_k + t\phi) = I_k(F_k + t\phi) = \frac{\lambda_k(F_k + t\phi)}{|F_k + t\phi|^2}$ , we get

$$D_{F_k} I_k(\phi) = \frac{d}{dt} I_k \circ (F_k + t\phi)|_{t=0} = \lambda_k \left( \frac{\phi}{|F_k|^2} - 2 \frac{F_k}{|F_k|^4} F_k \cdot \phi \right).$$

Then we have

$$\int_D \langle Q[F_k], d\phi \rangle dx = \frac{3}{2} \Lambda_k \int_D \left( \frac{1}{|F_k|^2} \langle S[f_k], \phi \rangle - 2 \frac{1}{|F_k|^4} \langle S[f_k], F_k \rangle \langle F_k, \phi \rangle \right) dx.$$

Recalling that  $F_k = \lambda_k \frac{f_k}{|f_k|^2}$ , we get

$$\operatorname{div} Q[F_k] = \frac{3}{2} \frac{\Lambda_k}{\lambda_k} (|f_k|^2 S[f_k] - 2\langle S[f_k], f_k \rangle f_k).$$

Since  $S^2 \setminus D_{t_k}$  is also conformal to  $D$ , we compute

$$\begin{aligned} - \int_{\partial D_{t_k}} \langle Q[F_k], \partial_r \rangle &= \int_{S^2 \setminus D_{t_k}} \operatorname{div} Q[F_k] \\ &= \frac{3}{2} \frac{\Lambda_k}{\lambda_k} \int_{S^2 \setminus D_{t_k}} (|f_k|^2 S[f_k] - 2\langle S[f_k], f_k \rangle f_k) \\ &= \frac{3}{2} \frac{\Lambda_k}{\lambda_k} \int_{S^2 \setminus D_{t_k}} e^{-2u_k} (|f_k|^2 S[f_k] - 2\langle S[f_k], f_k \rangle f_k) d\mu_{f_k}. \end{aligned}$$

Obviously,

$$\lim_{k \rightarrow +\infty} \int_{\partial D_{t_k}} \langle Q[F_k], \partial_r \rangle ds = \lim_{k \rightarrow +\infty} \int_{\partial D_1} \langle Q[I(\frac{f_k(t_k z)}{\lambda_k})], \partial_r \rangle d\theta = \int_{\partial D_1} \langle Q[I(f^2)], \partial_r \rangle d\theta.$$

We have

$$\int_{S^2 \setminus D_{t_k}} e^{-2u_k} (|f_k|^2 S[f_k] - 2\langle S[f_k], f_k \rangle f_k) d\mu_{f_k} = \int_{S^2 \setminus D_\delta} \cdots + \int_{D_\delta \setminus D_{t_k}} \cdots.$$

Since

$$|e^{-2u_k} (|f_k|^2 S[f_k] - 2\langle S[f_k], f_k \rangle f_k)| < C(\sigma(f_k)|H_{f_k}| + 1),$$

we have

$$\begin{aligned} &\left| \int_{D_\delta \setminus D_{t_k}} e^{-2u_k} (|f_k|^2 S[f_k] - 2\langle S[f_k], f_k \rangle f_k) d\mu_{f_k} \right| \\ &\leq C \left( \int_{D_\delta \setminus D_{t_k}} \sigma(f_k)|H_{f_k}| d\mu_{f_k} + \mu_{f_k}(D_\delta \setminus D_{t_k}) \right) \\ &\leq C \left( \sigma(f_k) \sqrt{W(f_k) \mu_{f_k}(D_\delta \setminus D_{\frac{r_k}{\delta}})} + \mu_{f_k}(D_\delta \setminus D_{\frac{r_k}{\delta}}) \right). \end{aligned}$$

By (2.1),

$$\lim_{\delta \rightarrow 0} \lim_{k \rightarrow +\infty} \int_{D_\delta \setminus D_{t_k}} e^{-2u_k} (|f_k|^2 S[f_k] - 2\langle S[f_k], f_k \rangle f_k) d\mu_{f_k} = 0.$$

By a direct calculation, on  $S^2 \setminus D_\delta$ , we have

$$\lim_{k \rightarrow +\infty} e^{-2u_k} (|f_k|^2 S[f_k] - 2\langle S[f_k], f_k \rangle f_k) = 4\sqrt{\pi}(n_{f_0}|f_0|^2 - 2f_0 \cdot n_{f_0}f_0).$$

Then

$$\lim_{\delta \rightarrow 0} \lim_{k \rightarrow +\infty} \int_{S^2 \setminus D_\delta} e^{-2u_k} (|f_k|^2 S[f_k] - 2\langle S[f_k], f_k \rangle f_k) d\mu_{f_k} = 4\sqrt{\pi} \int_{S^2} (n_{f_0}|f_0|^2 - 2f_0 \cdot n_{f_0}f_0) d\mu_{f_0}.$$

Let  $y_0$  be the center of  $f^1$ , which is nonzero because  $0 \in f^1(S^2)$ , and  $R = \frac{1}{\sqrt{8\pi}}$ . Then

$$\begin{aligned} \int_{S^2} (n_{f_0} |f_0|^2 - 2f_0 \cdot n_{f_0} f_0) d\mu_{f_0} &= \int_{S^2} (n_{f_0} |Rn_{f_0} + y_0|^2 - 2\langle Rn_{f_0} + y_0, n_{f_0} \rangle (Rn_{f_0} + y_0)) d\mu_{f_0} \\ &= \int_{S^2} (-R^2 n_{f_0} + |y_0|^2 n_{f_0} - 2Ry_0 - 2\langle y_0, n \rangle y_0) d\mu_{f_0} \\ &= -2Ry_0 \int_{S^2} d\mu_{f_0} = -Ry_0 \end{aligned}$$

where we used that

$$\int_{S^2} n_{f_0} = 0 \quad \text{and} \quad \int_{S^2} n_{f_0} \cdot y_0 = 0.$$

Hence finally

$$(5.4) \quad \int_{\partial D_1} \langle Q[I_0(f^2)], \partial_r \rangle = \frac{3}{2} \sqrt{2} y_0 \lim_{k \rightarrow +\infty} \frac{\Lambda_k}{\lambda_k}.$$

It is well-known that  $\int_{\partial D_1} \langle Q[I(f^2)], \partial_r \rangle$  is a nonzero vector parallel to the symmetry axis of  $f^2$  (cf [15] or [1]).

## 6. APPENDIX 1

In this appendix, we use results from [13] to derive estimates for the equation

$$(6.1) \quad \Delta_g H + |A^\circ|^2 H = aH + b, \quad \text{for } a, b \in \mathbb{R} \text{ constant.}$$

We start with higher order estimates when the curvature is not concentrated.

**Lemma 6.1.** *Let  $f : \Sigma \rightarrow \mathbb{R}^3$  be a properly immersed surface which satisfies (6.1) for  $|a| + |b| < \lambda$ . Let  $\Sigma_1 = f^{-1}(B_1(x_0)) \subset \subset \Sigma$ , and assume that  $\int_{\Sigma_1} |A|^2 < \epsilon_0$ . Then*

$$\|\nabla^m A\|_{L^\infty(\Sigma_{\frac{1}{2}})} < C = C(m, \lambda, \Lambda).$$

*Proof.* Applying Theorem 2.10 in [13] yields

$$\begin{aligned} \|A\|_{L^\infty(\Sigma_{\frac{\varrho}{2}}(x_0))} &\leq \frac{C}{\varrho} \|A\|_{L^2(\Sigma_\varrho(x_0))} + C\varrho(|a| \|H\|_{L^2(\Sigma_\varrho(x_0))} + |b| \mu(\Sigma_\varrho(x_0))^{\frac{1}{2}}) \\ &\leq \frac{C}{\varrho} \|A\|_{L^2(\Sigma_\varrho(x_0))} + C(|a|\varrho + |b|\varrho^2), \end{aligned}$$

where we used the quadratic area growth [20]. In particular

$$(6.2) \quad \|A\|_{L^\infty(\Sigma_{\frac{3}{4}})} \leq C.$$

Now each  $x \in f(\Sigma) \cap B_{\frac{1}{2}}(x_0)$  has finite preimage  $\{y^1, \dots, y^m\}$ . By Langer's theorem [16], for each  $\alpha \leq 1$  there is a constant  $r > 0$  such that  $f$  is an  $(r, \alpha)$ -immersion on  $\Sigma_{\frac{3}{4}}$ . This means that on suitable neighborhoods  $U^i$  of  $y^i$ , the  $f|_{U^i}$  are graphs over  $P^i \cap B_r(x)$ , where  $P^i = \text{im } Df(y^i)$  are the tangent planes, with graph functions  $u^i$  satisfying

$$\|\nabla u^i\|_{C^0(B_r(x) \cap P^i)} \leq \alpha.$$

In fact Langer shows in Lemma 2.1 in [16] that

$$\|u^i\|_{W^{2,p}(B_r(x) \cap P^i)} \leq C(p) \quad \text{for all } p < \infty.$$

It follows that the metric  $g_{\lambda\mu}$  satisfies in graph coordinates

$$\|g_{\lambda\mu}\|_{C^{0,\beta}} \leq C, \quad |g_{\lambda\mu} - \delta_{\lambda\mu}| \leq C\alpha.$$

For  $f(x, y) = (x, y, u(x, y))$  we now have the equations

$$(6.3) \quad g_{\lambda\mu} = \delta_{\lambda\mu} + \partial_\lambda u \partial_\mu u,$$

$$(6.4) \quad g^{\lambda\mu} \partial_{\lambda\mu}^2 H + \frac{\partial_\lambda(\sqrt{\det g} g^{\lambda\mu})}{\sqrt{\det g}} \partial_\mu H = -|A^\circ|^2 H + aH + b,$$

$$(6.5) \quad \left( \delta_{\lambda\mu} - \frac{\partial_\lambda u \partial_\mu u}{1 + |Du|^2} \right) \partial_{\lambda\mu}^2 u = H \sqrt{1 + |Du|^2},$$

$$(6.6) \quad A_{\lambda\mu}^3 = \left( 1 - g^{\gamma\nu} \partial_\gamma u \partial_\nu u \right) \partial_{\lambda\mu}^2 u,$$

From (6.2) and (6.6) we know that  $\nabla^2 u$  is bounded, and so is  $\nabla g_{\lambda\mu}$  by (6.3). Therefore we can apply  $L^p$ -theory to (6.4), see Theorem 9.11 in [5], to show  $H \in W^{2,p}$  locally for any  $p < \infty$ . The right hand side of (6.5) then belongs to  $C^{1,\alpha}$  for any  $\alpha < 1$ , and the coefficients of the equation are also in  $C^{1,\alpha}$ . Schauder estimates give  $u \in C^{3,\alpha}$ , which in turn yields  $g \in C^{2,\alpha}$  and  $A \in C^{1,\alpha}$ . Returning to (6.4) improves to  $H \in C^{3,\alpha}$ . The lemma follows by iterating Schauder estimates.  $\square$

The next lemma shows a relation between a conformal map and an extrinsic estimate.

**Lemma 6.2.** *Let  $f \in C^\infty(D, \mathbb{R}^3)$ ,  $f(0) = 0$ , with metric  $g = e^{2u} g_{euc}$ . Assume*

$$(6.7) \quad \int_D |A|^2 \leq 8\pi - \tau \quad \text{and} \quad |u| \leq \beta.$$

*Let  $C_\delta(f)$  be the component of  $f^{-1}(B_\delta(0))$  containing the origin. Then for any  $r > 0$  there exists  $\delta = \delta(\tau, \beta, r) > 0$  such that*

$$C_\delta(f) \subset D_r.$$

*Likewise, for any  $\delta > 0$  there exists  $\rho = \rho(\tau, \beta, \delta) > 0$  such that*

$$D_\rho \subset C_\delta(f).$$

*Proof.* If the first statement fails then we can find  $\delta_k \rightarrow 0$  and  $f_k \in C^\infty(D, \mathbb{R}^3)$  with (6.7), where  $f_k(0) = 0$  and  $g_{f_k} = e^{2u_k} g_{euc}$ , such that there exist

$$z_k \in C_{\delta_k}(f_k) \quad \text{with} \quad |z_k| \geq r.$$

By results in [6] and [11],  $f_k$  converges to some  $f \in W_{conf}^{2,2}(D, \mathbb{R}^3)$  locally weakly in  $W^{2,2}(D)$  and strongly in  $C^0(D, \mathbb{R}^3)$ . Now there are curves  $\gamma_k : [0, 1] \rightarrow D$  with  $\gamma_k(0) = 0$ ,  $\gamma_k(1) = z_k$  and  $|f_k(\gamma_k(t))| < \delta_k$  for all  $t \in [0, 1]$ . It follows that

$$\min_{z \in \partial D_\varrho} |f(z)| = 0 \quad \text{for all } \varrho \in (0, r].$$

By the local expansion for  $W^{2,2}$  immersions, we conclude that  $f$  is constant. But  $|Df|^2 \geq \frac{1}{2}e^{-2\beta} > 0$ , a contradiction.

If the second claim was not true, then there exist points  $z_k \notin C_\delta(f_k)$  such that  $z_k \rightarrow 0$ , where  $f_k \in C^\infty(D, \mathbb{R}^n)$  satisfies  $f_k(0) = 0$  and (6.7). Now we have either  $|f(z_k)| \geq \delta$ , or  $z_k$  is in another component of  $f^{-1}(B_\delta(0))$ . In the second we can find  $\lambda_k \in (0, 1)$  such that

$|f(\lambda_k z_k)| \geq \delta$ . Thus, for both cases, we have  $\lambda_k \in (0, 1]$  such that  $|f(\lambda_k z_k)| \geq \delta$ .

As above the sequence converges to a  $W^{2,2}$  conformal immersion  $f : D \rightarrow \mathbb{R}^3$ , locally uniformly in  $D$ . But then

$$0 = \lim_{k \rightarrow \infty} |f_k(0)| = \lim_{k \rightarrow \infty} |f_k(\lambda_k z_k)| \geq \delta,$$

which is again a contradiction.  $\square$

Applying the above lemma, we obtain the following:

**Proposition 6.3.** *Let  $f : D \rightarrow \mathbb{R}^3$  be a conformal immersion with metric  $g_f = e^{2u} g_{\text{euc}}$ , which satisfies (6.1) with  $|a| + |b| \leq \alpha$ . We assume  $|u| \leq \beta$ . There exists a constant  $\epsilon_0 > 0$ , such that if  $\int_D |A|^2 d\mu_f < \epsilon_0$ , then for any  $r < 1$  and  $m > 0$  and  $m > 0$*

$$\|\nabla^m f\|_{L^\infty(D_r)} \leq C(m, r, \alpha, \beta).$$

The proof uses a priori estimates similar to Lemma 6.1 and is omitted.

## 7. APPENDIX 2

**Theorem 7.1.** *Let  $f_k \in W^{2,2}(D, \mathbb{R}^n)$  satisfy*

- 1)  $\int_D |A_{f_k}|^2 d\mu_{f_k} < 4\pi - \tau$ .
- 2)  $f_k$  can be extended to a closed immersed surface  $f_k : \Sigma_k \rightarrow \mathbb{R}^n$  with

$$\int_{\Sigma_k} |A_{f_k}|^2 d\mu_{f_k} < \Lambda.$$

Take a curve  $\gamma : [0, 1] \rightarrow D$ , and set  $\lambda_k = \text{diam } f_k(\gamma)$ . Then we can find a subsequence of  $\frac{f_k - f_k(\gamma(0))}{\lambda_k}$  which converges weakly in  $W_{\text{loc}}^{2,2}(D)$  to an  $f_0 \in W_{\text{conf,loc}}^{2,2}(D, \mathbb{R}^n)$ . Furthermore, we can find an inverse  $I = \frac{y - y_0}{|y - y_0|^2}$  with  $y_0 \notin f_0(D)$  such that

$$\int_{\Sigma} (1 + |A_{I(f_0)}|^2) d\mu_{I(f_0)} < +\infty.$$

*Proof.* Put  $f'_k = \frac{f_k - f_k(\gamma(0))}{\lambda_k}$ ,  $\Sigma'_k = \frac{\Sigma_k - f_k(\gamma(0))}{\lambda_k}$ . We have two cases:

Case 1:  $\text{diam}(f'_k) < C$ . By inequality (1.3) in [20] with  $\rho = \infty$ ,  $\frac{\Sigma'_k \cap B_\sigma(\gamma(0))}{\sigma^2} \leq C$  for any  $\sigma > 0$ . Hence we get  $\mu(f'_k) < C$  by taking  $\sigma = \text{diam}(f'_k)$ . Then by Helein's convergence theorem [6, 11],  $f'_k$  converges weakly in  $W_{\text{loc}}^{2,2}(D)$ . Since  $\text{diam } f'_k(\gamma) = 1$ , the weak limit is not trivial.

Case 2:  $\text{diam}(f'_k) \rightarrow +\infty$ . We take a point  $y_0 \in \mathbb{R}^n$  and a constant  $\delta > 0$ , s.t.

$$B_\delta(y_0) \cap \Sigma'_k = \emptyset, \quad \forall k.$$

Let  $I = \frac{y - y_0}{|y - y_0|^2}$ , and

$$f''_k = I(f'_k), \quad \Sigma''_k = I(\Sigma'_k).$$

By conformal invariance of Willmore functional we have

$$\int_{\Sigma''_k} |A_{\Sigma''_k}|^2 d\mu_{\Sigma''_k} = \int_{\Sigma_k} |A_{\Sigma_k}|^2 d\mu_{\Sigma_k} < \Lambda.$$



Since  $\Sigma_k'' \subset B_{\frac{1}{2}}(0)$ , also by (1.3) in [20], we get  $\mu(f_k'') < C$ . Let

$$\mathcal{S}(\{f_k''\}) := \{p \in D : \lim_{r \rightarrow 0} \lim_{k \rightarrow +\infty} \int_{D(p)} |A_{f_k''}|^2 d\mu_{f_k''} \geq 4\pi\}.$$

Then  $f_k''$  converges weakly in  $W_{loc}^{2,2}(D \setminus \mathcal{S}(f_k''))$ .

Next, we prove that  $f_k''$  does not converge to a point by assumption. If  $f_k''$  converges to a point in  $W_{loc}^{2,2}(D \setminus \mathcal{S}(f_k''))$ , then the limit must be 0, for  $\text{diam}(f_k')$  converges to  $+\infty$ . By the definition of  $f_k''$ , we can find a  $\delta_0 > 0$ , such that  $f_k''(\gamma) \cap B_{\delta_0}(0) = \emptyset$ . Thus for any  $p \in \gamma([0, 1]) \setminus \mathcal{S}(f_k'')$ ,  $f_k''$  will not converge to 0. A contradiction.

Then we only need to prove that  $f_k'$  converges weakly in  $W_{loc}^{2,2}(D, \mathbb{R}^n)$ . Let  $f_0''$  be the limit of  $f_k''$  which is a branched immersion of  $D$  in  $\mathbb{R}^n$ . Let  $\mathcal{S}^* = f_0''^{-1}(\{0\})$ , which is isolate. First, we prove that for any  $\Omega \subset\subset D \setminus (\mathcal{S}^* \cup \mathcal{S}(\{f_k''\}))$ ,  $f_k'$  converges weakly in  $W^{2,2}(D, \mathbb{R}^n)$ : Since  $f_0''$  is continuous on  $\bar{\Omega}$ , we may assume  $\text{dist}(0, f_0''(\Omega)) > \delta > 0$ . Then  $\text{dist}(0, f_k''(\Omega)) > \frac{\delta}{2}$  when  $k$  is sufficiently large. Noting that  $f_k' = \frac{f_k''}{|f_k''|^2} + y_0$ , we get that  $f_k'$  converges weakly in  $W^{2,2}(\Omega, \mathbb{R}^n)$ .

Next, we prove that for each  $p \in \mathcal{S}^* \cup \mathcal{S}(\{f_k''\})$ ,  $f_k'$  also converges in a neighborhood of  $p$ . Let  $g_{f_k'} = e^{2u_k'} g_{euc}$ . Since  $f_k' \in W_{conf}^{2,2}(D_{4r}(p))$  with  $\int_{D_{4r}(p)} |A_{f_k'}|^2 d\mu_{f_k'} < 4\pi - \tau$  when  $r$  is sufficiently small and  $k$  sufficiently large, by the arguments in [11], we can find a  $v_k$  solving the equation

$$-\Delta v_k = K_{f_k'} e^{2u_k'}, \quad z \in D_r \quad \text{and} \quad \|v_k\|_{L^\infty(D_r(p))} < C.$$

Since  $f_k'$  converges to a conformal immersion in  $D_{4r} \setminus D_{\frac{1}{4}r}(p)$ , we may assume that

$$\|u_k'\|_{L^\infty(D_{2r} \setminus D_r(p))} < C.$$

Then  $u_k' - v_k$  is a harmonic function with  $\|u_k' - v_k\|_{L^\infty(\partial D_{2r}(p))} < C$ , then we get  $\|u_k'(z) - v_k(z)\|_{L^\infty(D_{2r}(p))} < C$  from the Maximum Principle. Thus,  $\|u_k'\|_{L^\infty(D_{2r}(p))} < C$ , which implies  $\|\nabla f_k'\|_{L^\infty(D_{2r})} < C$ . By the equation  $\Delta f_k' = e^{2u_k'} H_{f_k'}$ , and the fact that

$$\|e^{2u_k'} H_{f_k'}\|_{L^2(D_{2r})}^2 < e^{2\|u_k'\|_{L^\infty}} \int_{D_{2r}} |H_{f_k'}|^2 d\mu_{f_k'},$$

we get  $\|\nabla f_k'\|_{W^{1,2}(D_r)} < C$ . We complete the proof.  $\square$

## REFERENCES

- [1] Y. Bernard, T. Rivière: Singularity removability at branch points for Willmore surfaces. *Pacific J. Math.* **265** (2013), no. 2, 257–311.
- [2] K. Berndt, R. Lipowsky, U. Seifert: Shape transformation of vesicles: phase diagram for spontaneous-curvature and bilayer-coupling models, *Phys. Rev. A* **44** (1991), 1182–1202.
- [3] P. B. Canham: The minimum energy of bending as a possible explanation for the biconcave shape of the human red blood cell. *J. Theor. Biology* **26** (1970), 61–81.
- [4] J. Chen, Y. Li: Bubble tree of a class of conformal mapping & applications to Willmore functional, *Amer. J. Math.* **136** (2014), 1107–1154.
- [5] D. Gilbarg, N. Trudinger: Elliptic partial differential equations of second order, Second edition, Springer Grundlehren, 224. Springer-Verlag 1983.

- [6] F. Hélein: Harmonic maps, conservation laws and moving frames. Translated from the 1996 French original. With a foreword by James Eells. Second edition. Cambridge Tracts in Mathematics, 150. Cambridge University Press, Cambridge, 2002.
- [7] W. Helfrich: Elastic properties of lipid bilayers: theory and possible experiments. *Zeitschrift für Naturforschung C*, **28** (1973), 693–703.
- [8] A. Huber: On subharmonic functions and differential geometry in the large, *Comment. Math. Helv.* **32** (1957), 181–206.
- [9] J. Jost: Two-dimensional geometric variational problems, John Wiley & Sons, Chichester 1991.
- [10] L. Keller, A. Mondino, T. Rivière: Embedded surfaces of arbitrary genus minimizing the Willmore energy under isoperimetric constraint. *Arch. Rational Mech. Anal.* **212** (2014), 645–682.
- [11] E. Kuwert and Y. Li:  $W^{2,2}$ -conformal immersions of a closed Riemann surface into  $\mathbb{R}^n$ . *Comm. Anal. Geom.* **20** (2012), 313–C340.
- [12] E. Kuwert and R. Schätzle: Gradient flow for the Willmore functional. *Comm. Anal. Geom.* **10** (2002), 307–339.
- [13] E. Kuwert and R. Schätzle: The Willmore flow with small initial energy. *J. Differential Geom.*, **57** (2001), 409–441.
- [14] E. Kuwert and R. Schätzle: Removability of point singularities of Willmore surfaces, *Ann. of Math.* **160** (2004), 315–357.
- [15] E. Kuwert and R. Schätzle: Closed surfaces with bounds on their Willmore energy, *Ann. Sc. Norm. Super. Pisa Cl. Sci.* **11** (2012), 605–634.
- [16] J. Langer: A compactness theorem for surfaces with  $L^p$ -bounded second fundamental form. *Math. Ann.*, **270** (1985) 223–234.
- [17] R. Lipowsky: The conformation of membranes, *Nature* **349** (1991), 475–481.
- [18] T. Nagasawa and I. Takagi : Bifurcating critical points of bending energy under constraints related to the shape of red blood cells. *Calc. Var. Partial Differ. Equ.* **16**, (2003), 63–111.
- [19] T. Rivière: Analysis aspects of Willmore surfaces. *Invent. Math.* **174** (2008), 1–45.
- [20] L. Simon: Existence of surfaces minimizing the Willmore functional, *Comm. Anal. Geom.* **1** (1993), 281–326.
- [21] J. Schygulla: Willmore minimizers with prescribed isoperimetric ratio. *Arch. Rational Mech. Anal.*, **203** (2012), 901–941.
- [22] T. J. Willmore: Total Curvature in Riemannian Geometry, John Wiley & Sons, New York (1982).

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